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GROEP TRANSPORTFYSICA

Answers Examination "Physical Transport Phenomena" 3T320 of 3 November 2009

Problem 1.

a). This term represents the viscous dissipation due to sheer stresses in the fluid. The term occurs in the energy equation and in the mechanical energy balance equation.

The terms of the stress tensor are given by $\tau_{ij} = -\mu\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)$. Filling in the terms of the given flow field gives the following results:

$\tau_{11}=\mu b$, $\tau_{22}=\mu b$, $\tau_{33}=-2\mu b$, $\tau_{12}=\tau_{13}=\tau_{21}=\tau_{23}=\tau_{31}=\tau_{32}=0$.

For the tensor $(\nabla v)_{ij}=\partial v_j/\partial x_i$ one finds the following results:

$\partial v_x/\partial x=-\frac{1}{2}b$, $\partial v_y/\partial y=-\frac{1}{2}b$, $\partial v_z/\partial z=b$ and all other terms are zero. So follows to evaluate:

$$\underline{\underline{\tau}}:\nabla\underline{v} = \begin{pmatrix} \mu b & 0 & 0 \\ 0 & \mu b & 0 \\ 0 & 0 & -2\mu b \end{pmatrix} : \begin{pmatrix} -\frac{1}{2}b & 0 & 0 \\ 0 & -\frac{1}{2}b & 0 \\ 0 & 0 & b \end{pmatrix} = Trace\begin{pmatrix} -\frac{1}{2}\mu b^2 & 0 & 0 \\ 0 & -\frac{1}{2}\mu b^2 & 0 \\ 0 & 0 & -2\mu b^2 \end{pmatrix} = -3\mu b^2$$

b). $\nabla \cdot \underline{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = -\frac{1}{2}\left(y^{-\frac{1}{2}} + z^{-\frac{1}{2}}\right)t^{-\frac{1}{2}} + \frac{1}{2}\left(y^{-\frac{1}{2}}\right)t^{-\frac{1}{2}} + \frac{1}{2}\left(z^{-\frac{1}{2}}\right)t^{-\frac{1}{2}} = 0$

The flow is incompressible.

For $D\underline{v}/Dt$ one finds:

$$\frac{Dv_x}{Dt} = \frac{\partial v_x}{\partial t} + (\underline{v}\cdot\nabla)v_x = \tfrac{1}{4}t^{-\frac{3}{2}}x\left(y^{-\frac{1}{2}} + z^{-\frac{1}{2}}\right) + \tfrac{1}{4}t^{-1}x\left(y^{-\frac{1}{2}} + z^{-\frac{1}{2}}\right)^2 + \tfrac{1}{4}t^{-1}x\left(y^{-1} + z^{-1}\right)$$

$$\frac{Dv_y}{Dt} = \frac{\partial v_y}{\partial t} + (\underline{v}\cdot\nabla)v_y = -\tfrac{1}{2}t^{-\frac{3}{2}}y^{\frac{1}{2}} + \tfrac{1}{2}t^{-1}$$

$$\frac{Dv_z}{Dt} = \frac{\partial v_z}{\partial t} + (\underline{v}\cdot\nabla)v_z = -\tfrac{1}{2}t^{-\frac{3}{2}}z^{\frac{1}{2}} + \tfrac{1}{2}t^{-1}$$

c). This law is valid provided that the following conditions hold:

- law must be applied along a streamline
- viscous forces are negligible, so at high Reynolds numbers, and outside viscous boundary layers
- flow is incompressible
- gravity forces are the only body forces.

In the case of the circular cylinder it is not possible to stay on a streamline as point B lies in the wake. Due to vortices there are no steady defined streamlines. Besides that via the vortices and viscous dissipation occurs in the wake, that contradicts the assumptions of the Bernoulli equation, namely energy conservation. These viscous forces are also an issue here.

d). A formula for the solar constant is derived in example 16.4-1 in Bird, Stewart and Lightfoot. The solar constant is given by:

$$\text{Solar Constant} = \frac{\sigma T_S^4}{4}\left(\frac{D_s}{r_{12}}\right)^2$$

Filling in the data gives a value of,

$$\text{Solar Constant} = \frac{5.667 \bullet 10^{-8} \bullet 5762^4}{4}\left(\frac{1.384 \bullet 10^9}{228 \bullet 10^9}\right)^2 = 575 \text{ W/m}^2$$

This is the intensity of the sunlight that arrives on the part of the planet Mars facing the Sun. For the energy balance there is equilibrium between the absorbed amount of radiation that is incident from the Sun and the emitted radiation by the surface of Mars.
The energy balance is as follows:

$$I_0 a = e\sigma T_m^4$$

Here the parameter I_0 is the solar constant and a and e the absorption and emission coefficients of the surface of Mars and T_m the temperature of the surface of Mars. Under equilibrium conditions $a=e$ and filling in data gives a temperature of 317 K. This is of course the maximum temperature. More towards the poles of the planet the incoming amount of radiation per unit of area decreases rapidly with the cosine of the angle of the normal of the plane with the r_{12} vector that connects Mars and the Sun. It even decays faster near the poles where the partial shadowed area starts (the place where not all sunlight will reach Mars). In Wikipedia a temperature of 295 K is quoted as maximum. So the estimation with this simple radiation equilibrium is not too far off.

e). The heat transfer via radiation is given in the text of the excercise. The heat transfer via conduction follows from Fouriers law:

$$q_c = \frac{k}{d}(T_1 - T_2)$$

Both heat transfers can be set equal to each other giving:

$\sigma\left(T_1^4 - T_2^4\right) = \frac{k}{d}(T_1 - T_2)$, and this can be rewritten to:

$$\sigma(T_1 - T_2)(T_1 + T_2)\left(T_1^2 + T_2^2\right) = \frac{k}{d}(T_1 - T_2).$$

So,

$$(T_1 + T_2)\left(T_1^2 + T_2^2\right) = \frac{k}{d\sigma}.$$

Now using the suggestion as in the text one finds:

$$(T_1 + T_1 + \Delta T)\left(T_1^2 + (T_1 + \Delta T)^2\right) = \frac{k}{d\sigma}.$$

This is equal to:

$$T_1^3\left(2 + 2\frac{\Delta T}{T_1} + \left(\frac{\Delta T}{T_1}\right)^2\right)\left(2 + \left(\frac{\Delta T}{T_1}\right)\right) = \frac{k}{d\sigma}$$

Neglecting immediately the terms of order 2 between the brackets, and evaluating the product and again neglecting terms of order 2 gives:

$$4T_1^3\left(1 + \frac{3}{2}\frac{\Delta T}{T_1}\right) = \frac{k}{d\sigma}.$$

So the formula that is wanted is:

$$\Delta T = \left(\frac{k}{d\sigma}\frac{1}{4T_1^3} - 1\right)\frac{2}{3}T_1.$$

When filling in numbers one finds for $\Delta T=7.3$ K.
There are circumstances that radiation is always dominant namely when:

$$\frac{k}{d\sigma}\frac{1}{4T_1^3}<1 \,.$$

Problem 2.

a). The flow is stable, steady and developed. This means that comparing the flow profile in different z-cross sections it should be exactly the same. As the amount of mass flux inflow on the top of the plate is constant it follows immediately that the film thickness δ is constant. Some more argumentation follows from the fact that a thin stable film is considered, so from the deposition point on the top of the plate the film has the tendency to be accelerated by the gravity forces, however, viscosity and the shear stresses τ_{ij} will prevent it from accelerating unrestricted. So in the end there will be a force balance of gravity and viscous forces by which downflow always equals inflow.

b). The control volume that can be used is indicated in the diagram. There are six surfaces and taking the normals in the positive directions of the axes one finds for the momentum in the z-direction:

$$\left(\Phi_{xz}\big|_{x_0}-\Phi_{xz}\big|_{x_0+\Delta x}\right)WL+\left(\Phi_{yz}\big|_{y_0}-\Phi_{yz}\big|_{y_0+W}\right)L\Delta x+\left(\Phi_{zz}\big|_{z_0}-\Phi_{zz}\big|_{z_0+L}\right)W\Delta x+\rho g_z LW\Delta x=0$$

The second term for the plane normal to y is zero, because in the y-direction there is symmetry so momentum inflow equals outflow for both surfaces normal to y. For the other terms we have to rework them in more detail to know if they contribute or not. So:

$$\left(\Phi_{xz}\big|_{x_0}-\Phi_{xz}\big|_{x_0+\Delta x}\right)WL+\left(\Phi_{zz}\big|_{z_0}-\Phi_{zz}\big|_{z_0+L}\right)W\Delta x+\rho g_z LW\Delta x=0$$

Dividing by $WL\Delta x$ gives:

$$-\frac{\Delta\Phi_{xz}}{\Delta x}+\frac{\left(\Phi_{zz}\big|_{z_0}-\Phi_{zz}\big|_{z_0+L}\right)}{L}+\rho g_z=0$$

Taking the limit for $\Delta x\to 0$ gives:

$$\frac{\partial\Phi_{xz}}{\partial x}-\frac{\left(\Phi_{zz}\big|_{z_0}-\Phi_{zz}\big|_{z_0+L}\right)}{L}=\rho g_z$$

c). At the interface between film and air the condition for the stress tensor term $\tau_{xz}=0$ will hold, or $\partial v_z/\partial x=0$ at $x=\delta$. This is so because the viscosity of the air is much smaller than in the film so the stresses in the liquid film cannot be large. So there is no gradient in the fluid velocity v_z at the film-air interface, and the velocity more or less resembles the schematically drawn profile.
The second boundary condition is that at the wall at $x=0$ the no-slip condition holds or $v_z=0$.

d). One has to insert the components of the terms in the momentum balance and these are the following expressions holding for Newtons law of viscosity:

$$\Phi_{xz}=\tau_{xz}+\rho v_x v_z=-\mu\frac{\partial v_z}{\partial x}+\rho v_x v_z$$

$$\Phi_{zz}=\tau_{zz}+\rho v_z v_z=p-\mu\frac{\partial v_z}{\partial z}+\rho v_z v_z$$

So filling in leads to:

$$\frac{\partial\left(-\mu\frac{\partial v_z}{\partial x}+\rho v_x v_z\right)}{\partial x}-\frac{\left(\left(p-\mu\frac{\partial v_z}{\partial z}+\rho v_z v_z\right)\Big|_{z_0}-\left(p-\mu\frac{\partial v_z}{\partial z}+\rho v_z v_z\right)\Big|_{z_0+L}\right)}{L}=\rho g_z \ .$$

As is stated the velocity in the x-direction is zero, so all components with v_x in the expression are zero.
The pressure is everywhere p_0 so this means that the pressure term at z_0 will cancel the term at z_0+L
Because the profile is developed the terms with v_z will cancel by the same argument.
Furthermore as the profile is developed the gradient $\partial v_z/\partial z=0$.
The equation that remains now is:

$$\frac{\partial\left(-\mu\frac{\partial v_z}{\partial x}\right)}{\partial x}=\rho g_z \quad \text{or} \quad \frac{\partial^2 v_z}{\partial x^2}=-\frac{\rho g_z}{\mu} \ .$$

This is the requested equation.

e). The velocity profile can now be found by integration. Integrating once yields:

$$\frac{\partial v_z}{\partial x}=-\frac{\rho g_z}{\mu}x+C_1 \ .$$

Because of the boundary condition at the interface $x=\delta$ see part c). it immediately follows that $C_1=\rho g_z\delta/\mu$.
Integrating again yields:

$$v_z=-\frac{\rho g_z}{2\mu}x^2+\frac{\rho g_z\delta}{\mu}x+C_2$$

Now $v_z=0$ for $x=0$ and this gives directly that $C_2=0$.
The expression of the velocity profile follows as:

$$v_z=\frac{\rho g_z\delta^2}{2\mu}\frac{x}{\delta}\left(2-\frac{x}{\delta}\right)$$

f). The average velocity is given by integrating the velocity profile over the thickness of the film and dividing by then film thickness and yields the following expression:

$$\langle v_z\rangle=\frac{\int_{x=0}^{x=\delta} v_z dx}{\delta}=\frac{\int_{x=0}^{x=\delta}\frac{\rho g_z\delta^2}{2\mu}\frac{x}{\delta}\left(2-\frac{x}{\delta}\right)dx}{\delta}=\frac{\left(\frac{\rho g_z}{2\mu}\right)\delta^2\left(\frac{x^2}{\delta}-\frac{x^3}{3\delta^2}\right)\Big|_0^\delta}{\delta}$$

This gives for the average velocity:

$$\langle v_z\rangle=\frac{\rho g_z\delta^2}{3\mu} \ .$$

g). The basic mass balance for an elementary fluid element as indicated in figure 2a that has a width W in the y-direction is as follows:

$$\frac{\partial}{\partial t}(\rho\delta W\Delta z)=\left(\rho\langle v_z\rangle W\delta\right)_{z_0}-\left(\rho\langle v_z\rangle W\delta\right)_{z_0+\Delta z}$$

Here on the left hand side of the equality sign is the netto amount of change of mass of the fluid element. On the right hand side the netto influx and outflux into the element. Rearranging gives:

$$\frac{\partial \delta}{\partial t} = -\frac{\Delta(\delta\langle v_z\rangle)}{\Delta z}$$

Taking the limit for $\Delta z \rightarrow 0$ gives:

$$\frac{\partial \delta}{\partial t} = -\frac{\partial(\delta\langle v_z\rangle)}{\partial z}$$

This is the equation asked for.

h). Filling in the expression for $\langle v_z \rangle$ as answer of f). in the equation just derived gives:

$$\frac{\partial \delta}{\partial t} = -\frac{\partial}{\partial z}\left(\delta \frac{\rho g_z}{3\mu}\delta^2\right) \text{ so finally}$$

$$\frac{\partial \delta}{\partial t} = -\frac{\rho g_z}{\mu}\delta^2 \frac{\partial \delta}{\partial z}.$$

This is a quasilinear partial differential equation for δ that cannot easily be solved, however, a solution is given and can be checked by substitution into the differential equation.
Starting from,

$$\delta(z,t) = \frac{1}{\sqrt{\frac{\rho g_z}{\mu}}}\sqrt{\frac{z}{t}} \text{ it follows that}$$

$$\frac{\partial \delta}{\partial t} = -\frac{1}{2}\frac{1}{\sqrt{\frac{\rho g_z}{\mu}}} z^{\frac{1}{2}} t^{-\frac{3}{2}}$$

$$-\frac{\rho g_z}{\mu}\delta^2 \frac{\partial \delta}{\partial z} = -\frac{\rho g_z}{\mu}\frac{1}{\frac{\rho g_z}{\mu}}\frac{z}{t}\frac{1}{\sqrt{\frac{\rho g_z}{\mu}}}\frac{1}{2}z^{-\frac{1}{2}}t^{-\frac{1}{2}} = -\frac{1}{2}\frac{1}{\sqrt{\frac{\rho g_z}{\mu}}} z^{\frac{1}{2}} t^{-\frac{3}{2}}$$

So after filling in it is clear that both terms are equal and the given solution satisfies the differential equation. The solution shows that for large times the film is thinning, and for increasing z the thickness grows. This seems reasonable and is in agreement with experiments. It was found by Harold Jeffreys - Proc. Cambr. Phil. Soc. 26(1930)p.204

Problem 3.

a). Using the overall balance for the cylindrical wire and with the assumption that $v=0$ everywhere, energy conservation for the cylinder reduces to:

(Netto heat flux) + (Heat Generation) = 0

In a formula this is:

$$-2\pi a L q_r(a) + I^2 R = 0$$

Rewriting gives:

$$q_r(a) = \frac{RI^2}{2\pi a L}$$

b). Using the result of a). and equalizing to the heat flux density from the radiation at the outside one finds:

$$\sigma\left(T_w^4 - T_\infty^4\right) = \frac{I^2 R}{2\pi a L} \quad \Rightarrow \quad T_w = \sqrt[4]{\frac{I^2 R}{2\pi\sigma a L} + T_\infty^4}$$

c). The definition of ϕ is the heat production per unit of volume, so knowing the volume of the wire $\pi a^2 L$ it follows immediately that:

$$\phi = \frac{I^2 R}{\pi a^2 L} .$$

d). For a thin cylindrical shell in the wire holds the same law as stated in a). but now there is an influx as well as an outflux of heat. The result is:

$$\left(2\pi r q_r L\right)\Big|_{r=r} - \left(2\pi(r+dr) q_r L\right)\Big|_{r=r+dr} + \phi 2\pi r dr L = 0$$

or

$$d(rq_r) = \phi r dr \quad \Rightarrow \quad \frac{d(q_r r)}{dr} = \phi r \quad \Rightarrow \quad \frac{1}{r}\frac{d(q_r r)}{dr} = \phi .$$

The (middle) equation here above can be integrated and one finds:

$$q_r r = \tfrac{1}{2}\phi r^2 + C_1 \quad \Rightarrow \quad q_r = \tfrac{1}{2}\phi r + \frac{C_1}{r}$$

When r = 0 is filled in the heat flux should be finite, so $C_1 = 0$. The solution for q_r is as follows:

$$q_r(r) = \tfrac{1}{2}\phi r .$$

e). The result in e). can be combined with Fouriers law. This gives:

$$-k\frac{dT}{dr} = \tfrac{1}{2}\phi r .$$

Integrating this gives:

$$T(r) = -\frac{1}{4}\frac{\phi}{k} r^2 + C_2 .$$

The value of C_2 has to be found from the other boundary condition that was found in b), namely:

$$T(a) = T_w \quad \Rightarrow \quad -\frac{1}{4}\frac{\phi}{k}a^2 + C_2 = \sqrt[4]{\frac{I^2 R}{2\pi\sigma a L} + T_\infty^4}$$. This gives finally:

$$C_2 = \frac{1}{4}\frac{\phi}{k}a^2 + \sqrt[4]{\frac{I^2 R}{2\pi\sigma a L} + T_\infty^4} .$$

The final solution for *T(r)* is thus:

$$T(r) = \frac{1}{4}\frac{\phi}{k}\left(a^2 - r^2\right) + \sqrt[4]{\frac{I^2 R}{2\pi\sigma a L} + T_\infty^4}$$, and using the answer of c). it follows further:

$$T(r) = \frac{1}{4}\frac{\phi}{k}\left(a^2 - r^2\right) + \sqrt[4]{\frac{\phi a}{2\sigma} + T_\infty^4} .$$